$\mathcal{PT}$-symmetric interpretation of unstable effective potentials

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The conventional interpretation of the one-loop effective potentials of the Higgs field in the Standard Model and the gravitino condensate in dynamically broken supergravity is that these theories are unstable at large field values. A $\mathcal{PT}$-symmetric reinterpretation of these models at a quantum-mechanical level eliminates these instabilities and suggests the conjecture that these instabilities may also be tamed at the quantum-field-theory level.

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I. INTRODUCTION AND SUMMARY

Quantum field theory is an extremely complicated mathematical and physical construct. The dynamics takes place in an infinite-dimensional space and this means that analytic and numerical calculations are difficult. Approximation methods have been designed to make progress in understanding quantum field theory. One approach of historical significance is the use of an effective potential, which represents an infinite-dimensional theory by a function $V_{\rm eff}(\varphi_c)$ of a single variable φ_c called the classical field. The effective potential is then used to explore the physical attributes of the quantum field theory. Of course, $V_{\rm eff}(\varphi_c)$ is a severe approximation because it is only a one-dimensional approximation to an infinite-dimensional structure. This is the kind of approach that is used in the mini-superspace analysis of quantum cosmology [1].

As an illustration of a one-dimensional analysis of a difficult high-dimensional problem, let us consider a nuclear fission process. A fission process can be understood as a bag of several hundred nucleons that split into two smaller bags. Of course, the bag of nucleons undergoes violent quantum fluctuations and oscillations. Thus, a quantitative description of such a complicated process, either numerically or analytically, is almost intractable. However, we can approximate the process of fission by considering a bag of nucleons in static equilibrium. We then stretch this bag adiabatically into a dumbbell shape and plot the static potential of such a configuration as a one-dimensional function of the distance between the two lobes of the dumbbell. Having obtained a one-dimensional potential function, we can then make a one-dimensional semi-classical WKB approximation of the tunneling amplitude through the potential barrier. This calculation gives a reasonably accurate estimate the fission lifetime of a heavy nucleus [2].

It is in this spirit that we view an effective-potential approach in quantum field theory as a useful one-dimensional approximation to the full theory. The effective potential in quantum field theory is difficult to calculate and the standard perturbative method that is used is the loop expansion. Interestingly, when the effective potential for the Higgs field in the standard model is calculated in the one-loop approximation, $V_{\rm eff}(\varphi_c)$ becomes unbounded below for large values of the Higgs field. Many authors have taken this fact as an indication that the Higgs vacuum is unstable [3].

In this paper our approach will be to treat the effective potential as a function of the one-dimensional variable φ_c and to examine this potential using the techniques of quantum mechanics. We will argue that while an upside-down potential seems to suggest that the vacuum state is unstable, this may not actually be so. In particular, we will use the techniques of $\mathcal{PT}$-symmetric quantum mechanics to show that some quantum field theories, whose effective potentials suggest that the vacuum state of a theory is unstable, may actually possess a stable vacuum state.

Effective potentials have long been used to study symmetry breaking [4, 5] in field theory and much is known

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about the structure of such effective theories. The renormalized effective potential for a four-dimensional conformally invariant theory of a scalar field φ interacting with fermions and gauge fields has the form $\Gamma[\varphi]=\varphi^4 f\left[\log\left(\varphi^2/\mu^2\right),g\right]$, where μ is a mass scale and g denotes the coupling constants in the theory [6, 7]. Different theories are distinguished by the function f. The large-φ behavior of the effective action determines the stability of the vacuum state.

We consider two theories of current interest: (i) a theory of dynamical breaking of gravity via a gravitino condensate field φ, where [8, 9]

$$\Gamma[\varphi]\propto-\varphi^4\log(i\varphi)\quad(\varphi\text{ large});\tag{1}$$

(ii) the Standard Model of particle physics for which φ is the Higgs field [10] and

$$\Gamma[\varphi]\propto-\varphi^4\log(\varphi^2)\quad(\varphi\text{ large}).\tag{2}$$

Evidently, radiative corrections and renormalization can lead to effective potentials that suggest that the theory is unstable (has complex energy levels).

An early observation that renormalization can cause instability was made by Källén and Pauli [11], who showed that renormalizing the Lee model [12] makes the Hamiltonian complex and that the S-matrix becomes nonunitary because of ghost states. However, a $\mathcal{PT}$-symmetric analysis tames the apparent instabilities of the Lee model; the ghost states disappear, energies are real, and the S-matrix becomes unitary [13]. $\mathcal{PT}$-symmetric quantum theory also repairs the ghost problem in the Pais-Uhlenbeck model [14], the illusory instability of the double-scaling limit of O(N)-symmetric φ^4 theory [15, 16], and difficulties associated with the complex Hamiltonian for timelike Liouville field theory [17].

This article examines three $\mathcal{PT}$-symmetric quantum-mechanical Hamiltonians associated with the two problematic quantum field theories above. ($\mathcal{P}$ denotes parity reflection $x\to-x$, $p\to-p$; $\mathcal{T}$ denotes time reversal $x\to x$, $p\to-p$, $i\to-i$ [18].) The first Hamiltonian,

$$H=p^2+x^4\log(ix),\tag{3}$$

is a toy model we developed to study logarithmic $\mathcal{PT}$-symmetric theories. We show that the spectrum of this complex $\mathcal{PT}$-symmetric Hamiltonian is discrete, real, and positive. The second Hamiltonian,

$$H=p^2-x^4\log(ix),\tag{4}$$

is the quantum-mechanical analog of (1). We show that the spectrum of this complex and apparently unstable Hamiltonian is also discrete, real, and positive, and this suggests that there is no instability in the supergravity theory of inflation in Ref. [9]. The third Hamiltonian,

$$H=p^2-x^4\log(x^2),\tag{5}$$

is motivated by the renormalized effective potential for the Higgs model (2). We show that the ground-state energy of this Hamiltonian is real and positive, and this suggests the intriguing possibility that, contrary to earlier work [19], the Higgs vacuum may be stable.

These three models all have a new $\mathcal{PT}$-symmetric structure that has not previously been examined, namely, the logarithm term in the Hamiltonian. We show that the $\mathcal{PT}$ symmetry of the Hamiltonians (3) and (4) is *unbroken*; that is, their spectra are entirely real. However, the $\mathcal{PT}$ symmetry of H in (5) is *broken*; only the four lowest-lying states have real energy. Thus, while the ground state is stable, almost all other states in the theory are unstable. This is indeed what is observed in nature; almost all particles are unstable and there are only a few stable particles. This suggests the conjecture that the Higgs vacuum is stable as a consequence of $\mathcal{PT}$ symmetry and that the universe may be described by a Hamiltonian having a broken $\mathcal{PT}$-symmetry.

The structure of this paper is as follows: In Sec. II we discuss the toy Hamiltonian (3) and demonstrate that the energy eigenvalues are real as a consequence of the closed nature of the classical trajectories in the complex plane. In Sec. III we repeat the analysis for the Hamiltonian (4) with similar conclusions regarding the reality of the energy eigenvalues. Next, in Sec. IV we study the mini-superspace-like approximation to the one-loop effective Higgs potential. In this case the classical trajectories are open, which suggests that the quantum spectrum is not entirely real. Nevertheless, the lowest four energy eigenvalues are real. The reality of the lowest energy level suggests that the Higgs vacuum is actually stable. Our conclusions are presented in Sec. V.

II. ANALYSIS OF THE TOY MODEL HAMILTONIAN (3)

To analyze H in (3) we first locate the complex turning points. Next, we examine the complex classical trajectories on an infinite-sheeted Riemann surface and find that all these trajectories are closed. This shows that the energy

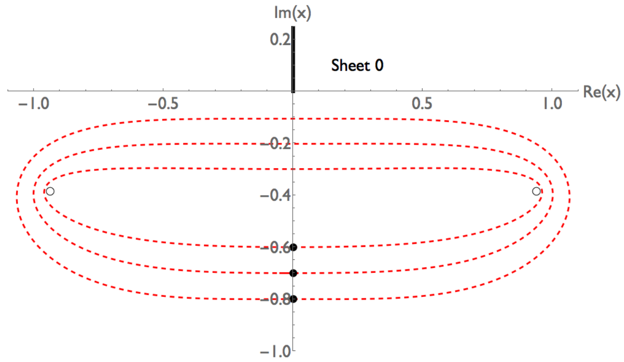

FIG. 1: [Color online] Three nested closed classical paths $x(t)$ (red dashed curves) on the principal sheet (sheet 0) of the Riemann surface plane for energy $E = 1.24909$ and initial conditions $x(0) = -0.6i$, $-0.7i$, $-0.8i$ (black dots). The paths do not cross the branch cut on the positive-imaginary axis (solid black line) so they remain on sheet 0. The paths are $\mathcal{PT}$ symmetric (left-right symmetric). Turning points at $\pm 0.938 - 0.385i$ (small circles) cause the paths to turn around in the right-half and left-half plane.

levels are all real [20]. We then perform a WKB calculation of the eigenvalues and note that the results agree with a precise numerical determination of the eigenvalues.

The turning points for H in (3) satisfy the equation

$$E = x^4 \log(ix), \tag{6}$$

where E is the energy. We take $E = 1.24909$ because this is the numerical value of the ground-state energy obtained by solving the Schrödinger equation for the potential $x^4 \log(ix)$ (see Table I).

One turning point lies on the negative imaginary-x axis. To find this point we set $x = -ir$ $(r > 0)$ and obtain the algebraic equation $E = r^4 \log r$. Solving this equation by using Newton's method, we find that the turning point lies at $x = -1.39316i$. To find the other turning points we seek solutions to (6) in polar form $x = re^{i\theta}$ $(r > 0,\, \theta\,\text{real})$. Substituting for x in (6) and taking the imaginary part, we obtain

$$\log r = -(2k\pi + \theta + \pi/2)\cos(4\theta)/\sin(4\theta), \tag{7}$$

where k is the sheet number in the Riemann surface of the logarithm. (We choose the branch cut to lie on the positive-imaginary axis.) Using (7), we simplify the real part of (6) to

$$E = -r^4(2k\pi + \theta + \pi/2)/\sin(4\theta). \tag{8}$$

We then use (7) to eliminate r from (8) and use Newton's method to determine θ. For $k = 0$ and $E = 1.24909$, two $\mathcal{PT}$-symmetric (left-right symmetric) pairs of turning points lie at $\pm 0.93803 - 0.38530i$ and at $\pm 0.32807 + 0.75353i$. For $k = 1$ and $E = 1.24909$ there is a turning point at $-0.53838 + 0.23100i$; the $\mathcal{PT}$-symmetric image of this turning point lies on sheet $k = -1$ at $0.53838 + 0.23100i$.

The turning points determine the shape of the classical trajectories. Two topologically different kinds of classical paths are shown in Figs. 1 and 2. All classical trajectories are *closed* and *left-right symmetric*, and this implies that the quantum energies are all real [20].

The WKB quantization condition is a complex path integral on the principal sheet of the logarithm $(k = 0)$. On this sheet a branch cut runs from the origin to $+i\infty$ on the imaginary axis; this choice of branch cut respects the $\mathcal{PT}$ symmetry of the configuration. The integration path goes from the left turning point $x_{\rm L}$ to the right turning point $x_{\rm R}$ [21]:

$$\left(n + \tfrac{1}{2}\right)\pi \sim \int_{x_{\rm L}}^{x_{\rm R}} dx \sqrt{E - V(x)} \quad (n >> 1). \tag{9}$$

If the energy is large $(E_n \gg 1)$, then from (7) with $k = 0$ we find that the turning points lie slightly below the real axis at $x_{\rm R} = re^{i\theta}$ and at $x_{\rm L} = re^{-\pi i - i\theta}$ with

$$\theta \sim -\pi/(8 \log r) \quad \text{and} \quad r^4 \log r \sim E. \tag{10}$$

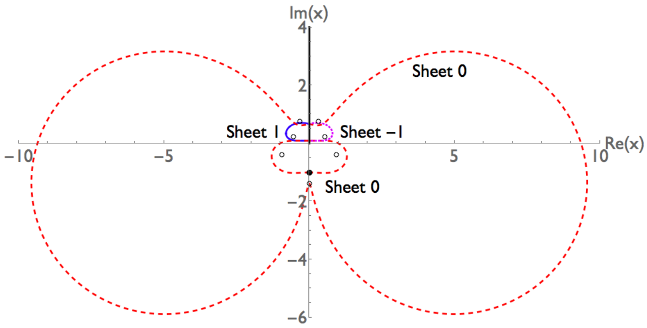

FIG. 2: [Color online] *Closed* classical path for energy $E = 1.24909$. The path begins at $x = -i$ (black dot) on sheet 0 of the logarithmic Riemann surface, moves to the right as a red dashed curve, and visits sheets -1 and 1 before returning to its starting point. The path is shown as a solid blue curve on sheet 1 and a dotted purple line on sheet -1. Turning points (small circles) at $x = \pm 0.938 - 0.385i$, $\pm 0.538 + 0.231i$, $\pm 0.328 + 0.754i$, and $-1.393i$ determine the shape of the path. The total path is $\mathcal{PT}$ symmetric (left-right symmetric).

We choose the path of integration in (9) to have a constant imaginary part so that the path is a horizontal line from $x_{\rm L}$ to $x_{\rm R}$. Since E is large, r is large and thus θ is small. We obtain the simplified approximate quantization condition

$$\left(n+\tfrac{1}{2}\right)\pi \sim r^3 \log r \int_{-1}^{1} dt \sqrt{1-t^4}, \tag{11}$$

which leads to the WKB approximation for $n \gg 1$:

$$\frac{E_n}{[\log(E_n)]^{1/3}} \sim \left[\frac{\Gamma(7/4)(n+1/2)\sqrt{\pi}}{\Gamma(5/4)\sqrt{2}}\right]^{4/3}. \tag{12}$$

To test the accuracy of (12) we have computed numerically the first 13 eigenvalues by solving the Schrödinger equation for (3). Some of these eigenvalues are listed in Table I. The accuracy of this WKB approximation increases smoothly with increasing n.

n	Numerical value of E_n	$\frac{E_n}{[\log(E_n)]^{1/3}}$	WKB prediction	% error
0	1.24909	2.06161	0.54627	73.5028 %
3	13.7383	9.96525	7.31480	26.5969 %
6	31.6658	20.9458	16.6979	20.2804 %
9	52.9939	33.4674	27.6956	17.2463 %
12	76.9748	47.1776	39.9324	15.3573 %

TABLE I: Eigenvalues of the Hamiltonian in (3) compared with the WKB approximation in (12).

III. ANALYSIS OF THE SUPERGRAVITY MODEL HAMILTONIAN (4)

The classical trajectories for the Hamiltonian (4) are plotted in Figs. 3 and 4. Like the classical trajectories for the Hamiltonian (3), these trajectories are closed, which implies that all the eigenvalues for H in (4) are real.

To find the eigenvalues of the complex Hamiltonian (4) we follow the procedure described in Ref. [18]; to wit, we obtain (4) as the parametric limit $\epsilon : 0 \to 2$ of the Hamiltonian $H = p^2 + x^2(ix)^\epsilon \log(ix)$, which has real positive eigenvalues when $\epsilon = 0$. As $\epsilon \to 2$, the Stokes wedges for the time-independent Schrödinger eigenvalue problem rotate into the complex-x plane [18]. Thus, this procedure defines the eigenvalue problem for H in (4) and specifies the energy levels. In Fig. 5 we plot the eigenvalues as functions of ϵ. Note that this figure is topologically identical to Fig. 1 in Ref. [18] except that the ground-state energy diverges at $\epsilon = -2$ rather than at $\epsilon = -1$. The spectrum

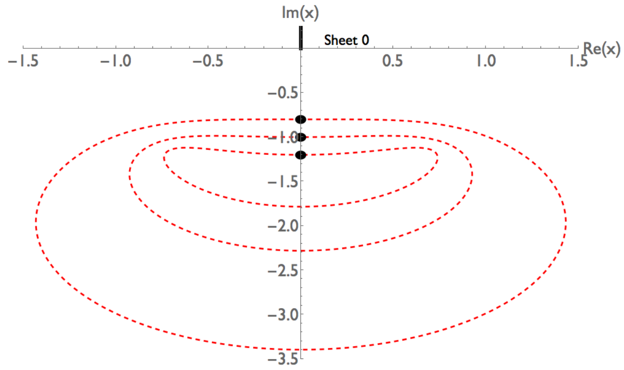

FIG. 3: [Color online] Three nested classical trajectories for H in (4) with $E = 2.07734$. The trajectories begin at $-0.8i$, $-i$, $-1.2i$ (black dots), do not cross the branch cut on the positive-imaginary axis (solid black line), and are closed.

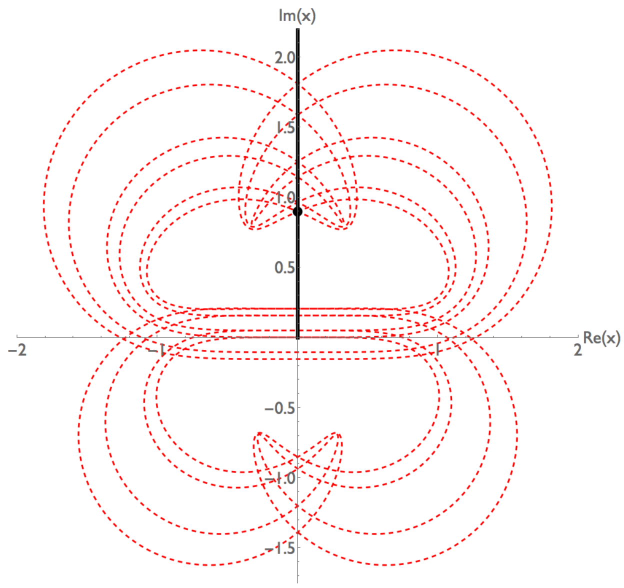

FIG. 4: [Color online] Complex classical trajectory for H in (4) with $E = 2.07734$. The trajectory begins at $0.9i$, crosses the branch cut on the positive-imaginary axis (solid black line), and visits three sheets of the Riemann surface but it is still closed and $\mathcal{PT}$ symmetric (left-right symmetric).

of $H = p^2 - ix$ is null [22]. This is because there are precisely three Stokes wedges of angular opening 120°. If the solution to the Schrödinger equation vanishes exponentially in one wedge, it grows exponentially in the adjacent two wedges and thus no eigenvalue condition is possible. The branch cut allows the Hamiltonian $H = p^2 - ix\log(ix)$ to evade this constraint; there are infinitely many Stokes wedges on the Riemann surface. Figure 1 indicates that when $\epsilon < 0$ the $\mathcal{PT}$ symmetry is broken, but that when $\epsilon \geq 0$ the $\mathcal{PT}$ symmetry is unbroken (all real eigenvalues).

WKB theory gives a good approximation to the eigenvalues of H in (4). We seek turning points for H in polar form $x = re^{i\theta}$ and find that on the principal sheet of the Riemann surface a $\mathcal{PT}$-symmetric pair of turning points lies at $\theta = -\pi/4 - \delta$ and $\theta = -3\pi/4 + \delta$. When $E \gg 1$, δ is small, $\delta \sim \pi/(16\log r)$, and r is large, $r^4 \log r \sim E$. The WKB calculation yields a formula for the eigenvalues that is identical to (12) except that there is no factor of $\sqrt{2}$ in the denominator. Thus, for large n the nth eigenvalue of H in (4) agrees approximately with the nth eigenvalue of H in (3) multiplied by $2^{2/3}$. A numerical determination of the first six eigenvalues gives 2.07734, 7.9189, 15.4216, 24.0932, 33.7053, and 44.1189.

IV. ANALYSIS OF THE HIGGS MODEL HAMILTONIAN (5)

To make sense of the Hamiltonian (5) we again introduce a parameter ϵ and we define H in (5) as the limit of $H = p^2 + x^2(ix)^\epsilon \log\left(x^2\right)$ as $\epsilon : 0 \to 2$. This case is distinctly different from that for H in (4). Figure 6 shows that the $\mathcal{PT}$ symmetry is broken for all $\epsilon \neq 0$. When $\epsilon = 2$, there are only four real eigenvalues: $E_0 = 1.1054311$, $E_1 = 4.577736$, $E_2 = 10.318036$, and $E_3 = 16.06707$. To confirm this result we plot a classical trajectory for $\epsilon = 2$ in Fig. 7. In contrast with Fig. 4, the trajectory is open and not left-right symmetric.

This result suggests that the Higgs vacuum is stable and that perhaps the real world is in a broken $\mathcal{PT}$-symmetric regime. This possibility has interesting implications for the $\mathcal{C}$ operator in $\mathcal{PT}$-symmetric quantum theory. In an unbroken regime the $\mathcal{C}$ operator, which is used to construct the Hilbert-space metric with respect to which the Hamiltonian is self-adjoint, commutes with the Hamiltonian and thus it cannot serve as the charge-conjugation operator in particle physics. However, in a broken $\mathcal{PT}$ regime, the states of H are not states of $\mathcal{C}$, and thus $\mathcal{C}$ may play the role of charge conjugation in particle physics [23].

V. CONCLUSIONS

In this article, we have discussed two types of field theoretic Hamiltonians that arise in interesting quantum-field-theory models. These are associated with supergravity theories having dynamically broken local supersymmetry and with the Higgs sector of the Standard Model of particle physics. The one-loop effective potential of such theories exhibits an upside-down behavior or has imaginary parts, which in the conventional treatment is interpreted as an instability of the quantum vacuum.

In the context of a mini-superspace-like approximation, both effective potentials have been shown to correspond to $\mathcal{PT}$-symmetric theories, the Higgs theory being characterized by broken $\mathcal{PT}$ symmetry. In the former case, the classical trajectories characterizing the solution of the one-dimensional Schrödinger equation are closed, which implies the reality of the entire spectrum of the energy eigenvalues. In the Higgs case, the trajectories are open, and only the lowest few eigenvalues are real. These results suggest that the vacuum in both cases is stable, but a detailed proof of this conjecture constitutes a long term challenge.

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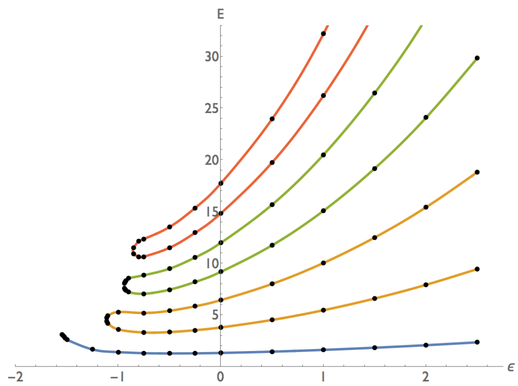

FIG. 5: [Color online] Energies of the Hamiltonian $H = p^2 + x^2(ix)^\epsilon \log(ix)$ plotted versus ϵ. This Hamiltonian reduces to H in (4) when $\epsilon = 2$. The energies are real when $\epsilon \geq 0$.

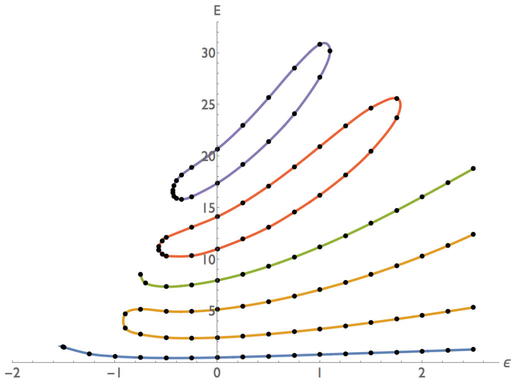

FIG. 6: [Color online] Eigenvalues of the Hamiltonian $H = p^2 + x^2(ix)^\epsilon \log\left(x^2\right)$, which reduces to H in (5) when $\epsilon = 2$. There are just four real energies when $\epsilon = 2$.

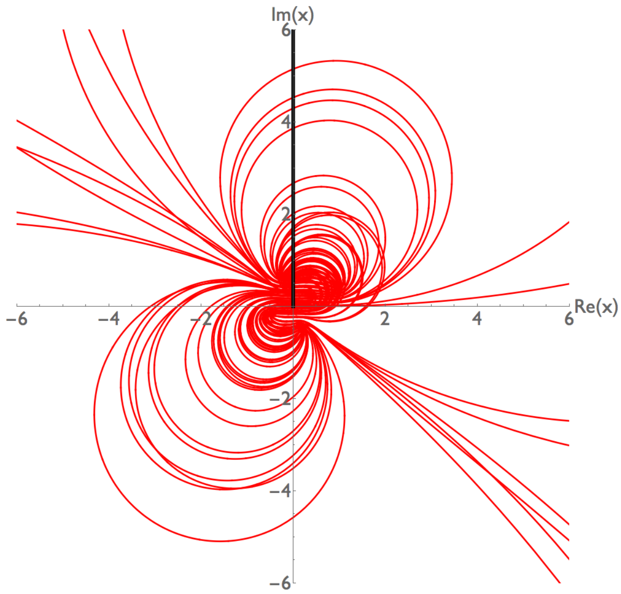

FIG. 7: [Color online] Classical path for the Hamiltonian $H = p^2 - x^4 \log\left(x^2\right)$. The initial point is $0.9i$ and the energy is $E = 1.10543$. The trajectory is not $\mathcal{PT}$ symmetric. It makes bigger and bigger loops and does not close.

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